Notes for Problems Worth Puzzling Through

NCTM, Boston, April 16, 2015

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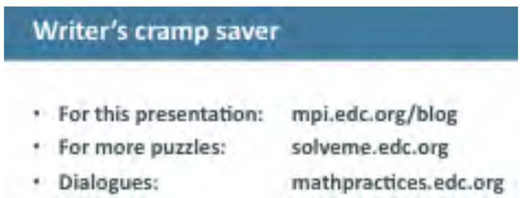

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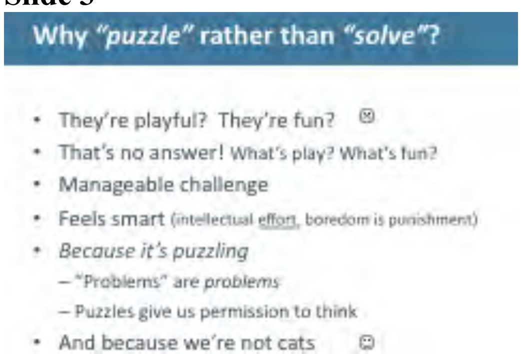


The strand was called "Problems worth *solving*." Why did we switch our title to "Problems worth puzzling through"?

Puzzles *are* playful and fun, but that's a circular answer because we're left asking *why* they are fun.

Like other animals, our brains are wired to give us a little "oooh, right, do that again" message when we do the things that sharpen the skills that let our species survive. That "do it again" message feels good, so we do it again. We get that message when we take on manageable challenges—efforts that are hard enough, but not so hard that we get a "ewww, no, don't do *that* again" message. Children, especially, push themselves to walk on lines, skip cracks, hop on one foot as far as they can go, ride bicycles, practice catching balls, run as fast as they can and, in general, do all sorts of physical things in the hardest possible ways. And, unless the hurt is too great, they keep trying this stuff even when it hurts, because the "do it again" message is stronger than the "ouch" message. They're building skills that our ancestors needed in order to tear through the brush to escape the tiger while, ourselves, trying to blow dart the rabbit that's trying to escape us.

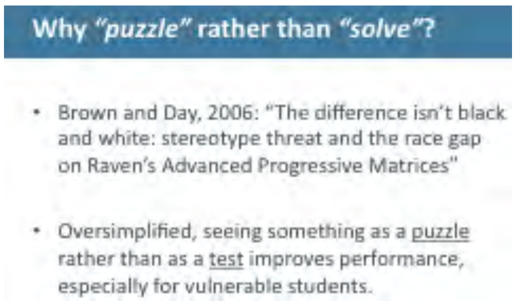
We also take on intellectual challenges, because keeping our brain sharp is more important to our species than keeping our teeth or nails sharp. Solving a puzzle makes us feel smart, and we like that. In fact, when we are bored, we will do *anything* to get out of it, including mischief we know will bring us a bit of trouble. Boredom hurts way more than a scraped knee.

And "problems" are a bit threatening. We are used to expecting that when we are given a problem in school, we're supposed to know pretty much immediately how to solve it, and if we don't see how, we feel there's something wrong with us—we didn't learn it properly (a not-too-bad "growth mindset," but still apicture of it being somehow *our* weakness) or, far, far worse, we are just no good at mathematics (the “fixed mindset” that gets us to avoid, because they are painful, other situations that might let us become better at mathematics). (See Carol Dweck’s research growth and fixed mindsets.)

By contrast, we expect *not* to know instantly how to solve a puzzle; that’s what *makes* it a puzzle. We give ourselves a bit more time to think, to puzzle it out.

If we were cats, we would scratch to keep our claws sharp, and pounce and chase to keep our eyes and reflexes sharp. But we’re people, so we like to keep our minds sharp. Though not everyone loves puzzles, they’re popular enough not to lurk only in dark corners of a specialty book store: even supermarkets have puzzle books for impulse purchases while you wait on line (bored!) at the checkout counter.

Slide 4



Raven’s Matrices was designed in 1936 with the intention of being an unbiased (culture-free, language-free) test of cognitive ability. Each item is a visual puzzle showing sequences of abstract designs that change in a systematic way. One sequence offers multiple choices about “what comes next” in an incomplete sequence.

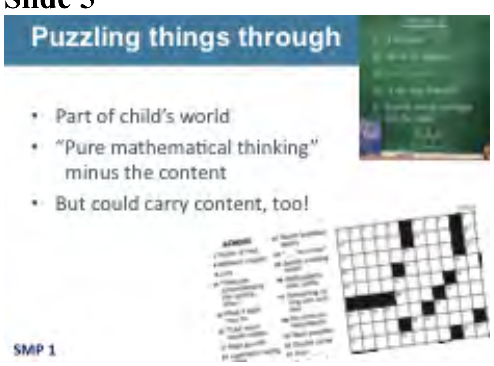
Because it does not depend on language or (presumably) any learned cultural knowledge, it was assumed to be the perfect research tool for asking nature/nurture questions about intelligence.

Because early results showed that African Americans performed worse on this test than whites, the obvious—but, it turns out, *wrong*—first conclusion was that there was a genuine difference in cognitive ability. There is (and even was, then) plenty of reason to believe that this is not true—there is *no* native difference in intelligence—but the Raven’s case needed an explanation. The 2006 Brown and Day study (a full 70 years after the invention of the test, though there were already a small number of studies just a very few years earlier)showed that there was, in fact, no difference between the two groups—black and white—in their ability to solve the puzzles on the Raven’s test. The difference had to do with the *threat* of testing. Presented *as a test*, the groups performed differently. But when the two groups were given the exact same problems to solve—the exact same items presented in the exact same order and in the exact same way—but those problems were described not as “an IQ test” but as “a set of puzzles…that the researchers wanted their opinions of,” the difference vanished. The two groups were equally “intelligent,” but the scores of the African American group were significantly lower when the threat was greater.

Two messages: One is clearly enormously important for the “achievement gap.” Many factors contribute, including (of course) poverty. But a huge one is low expectation of marginalized groups (by race or gender or economics), *including the groups’ own low self expectations*, instilled by constant low-expectation messages from the larger society. And another related one, is the threat of the situation. Tests are *tests*. Threatening. Puzzles are like all the challenges that kids take on, from roller-skating to hitting or throwing a ball accurately; they may be hard, even very hard, but not threats.

Get a description in http://en.wikipedia.org/wiki/Raven%27s_Progressive_Matrices and see a practice item at http://www.ravensprogressivematrices.com/

Slide 5



Puzzles are part of a child’s world, far more real-world to them than figuring out the cost of carpeting a room or how many marbles “Miriam” has.

Puzzling is also an important and basic habit of mind. Certain aspects of puzzling are inherently mathematical even when the puzzle has no mathematical content at all.

For example, in a crossword puzzle (no mathematical content), I start out *expecting* that of the hundred or so clues, most will be really useless at first, and that I have to scout around for the few places that might let me get started. *And*, after finding a possible entry point and making a guess about the word that fits that clue, I know I should look for clues for words that *cross* my guessed word to get a bit more evidence before I write my guesses in pen. If the “Down” clue *doesn’t* fit with my “Across” guess, I reevaluate both before writing in my guesses. This is the very heart of Standard 1 in the Common Core State Standards for Mathematical Practice: “Mathematically proficient students start by…looking for entry points…. They analyze…constraints, relationships….” “[They] check…using a different method” and “They…change course if necessary.”

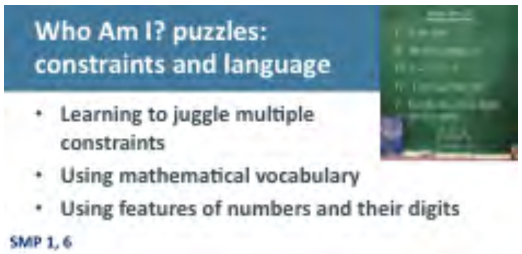
That's the way real life problems *mostly* are --- many possible starting places, and many possible clues, but we have to figure out which ones are actually useful *now* and which must wait for later.

The puzzles that we deliberately craft for entertainment are entertaining precisely *because* they are so much like life's real problems (but with no immediate threat—you don't starve or get eaten or run over or have your house collapse if you don't solve a crossword puzzle).

So, puzzles are real-life-like *and* very mathematical *in their nature*, even if they have no *content* that is mathematical. Crossword puzzles have no mathematical content, but involve the kind of "mathematical" thinking --- the puzzling-it-through disposition and looking for good entry points that CCSSM Standard 1 for Mathematical Practice calls for --- but puzzles can be crafted to carry mathematical *content* as well as the mathematical habits of mind.

The Who Am I? puzzle is one example.

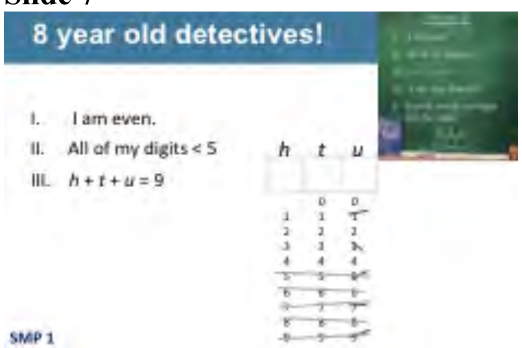
Slide 6



You could imagine writing such a puzzle on the board before your children arrive, and leaving it just as "entertainment"—a puzzle to solve.

Here is some of the content it can carry.

Slide 7



Let's solve it.

On the board in class, you have choices we don't have here. Take the clues in whatever order the kids lead. In an electronic presentation, everything comes in a fixed order.

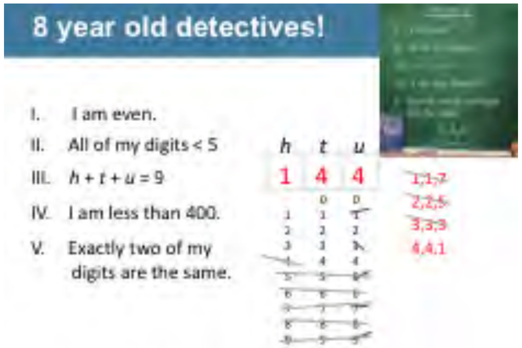
Clue 1: I could just write the 0, 2, 4, 6, 8 under the *u* but, for now, have chosen to write *all* the possible digits and cross out the ones that don't work. In class, I would not push for this, but would follow whatever lead is taken by the kids. If they don't know where to start, I might be tempted to use the fewer pencil marks, just writing what *can* go in that column.

Clue 2: Yay! I can cross out some things.

Clue 3: Hmmm…. I *could* make a list of all the possible ways of making 9 now, but maybe I'll just ignore this clue for a moment and see what else I can do.

Commentary: In "Who Am I?" puzzles, NOBODY ever wonders what the symbols mean, nor do they get shocked that two "different letters" could have the same "value" as in this case. These aren't "algebraic variables" to the kids, and don't *have* "values." The h, t, and u, are the obvious shorthand for what would otherwise take space and time to write. They are just a shorthand for English, which is the appropriate way to view them at this developmental stage. It is much easier to write h + t + u = 9 than to write "the hundreds digit plus the tens digit…" And there is nothing at all strange that two digits of a number might be the same. Nor do they need to learn that within a problem "t" always has (what *we* call) the same value, or that from problem to problem that value could change.

Slide 8



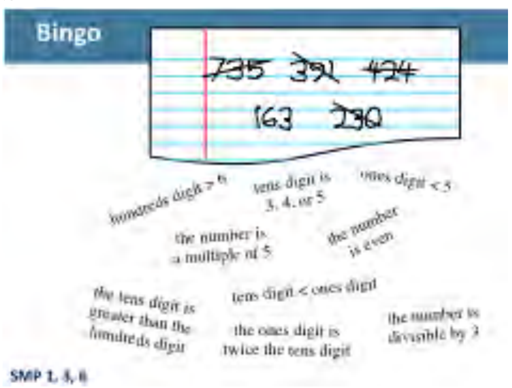
Clue 4: Great! I don't get to narrow down a lot, but I can narrow down a little!

Clue 5: Hmmmm…. If the two digits that are the same are 1, I'd need 1 + 1 + **7** to satisfy the third clue (my digits must add up to 9) and I don't have a 7. I don't have a 5, either. And if the two numbers that are the same are both 3, I'd need a third 3, and I don't have it (and, besides, *exactly two* of my digits are the same, not all three of them). So my digits must be 4, 4, and 1. But clue IV tells me that they're not in that order!

Solved! And I feel so smart solving it!!! ***And we want that feeling in our students!*** We *want* them to feel that they can be smart about mathematics. Nobody—and, for that matter, no corporation—undertakes effort in places where they expect not to succeed. We take risks if we feel we have a chance, and even put in effort if we think the result might reward us, but if we *expect* to fail, we put our efforts elsewhere. That is one reason why the children who need practice most are the same ones least inclined to do that practice.

Commentary: By the way, why did we number the clues I, II, III, IV, and V instead of with numbers? Because some child will then ask, and we can say (with no more fanfare than this) "Oh, this is how they wrote numbers in ancient Rome" and then, pointing, "One, two, three, four, five." Some child will be more curious, and the discussion could last another 10 seconds or so. Roman numerals are not important enough to take time away from other mathematical topics, but *curiosity* ***is*** important! If someone asks why "IV" is 4, you can treat that simply, too: "It is 1 before 5."

Slide 9

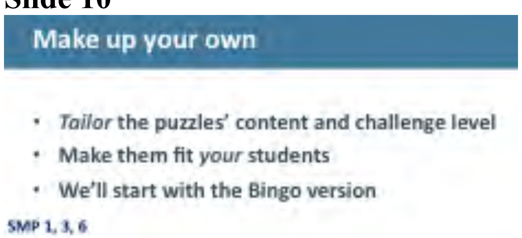


A class game version: Each player makes up five three-digit numbers. They take turns drawing a card with a "clue" (e.g., "hundreds digit > 6") on it. **All** players then cross out *one* number (if they have any) that matches that clue. When one has crossed out all numbers, they *check against the clues, justifying that they correctly used* them. Then they can make up a new set of numbers and play again.

Learning the *language* of the clues is the hardest part of this game, but children play repeatedly, and work out this language together. Both the Who Am I? puzzle and this bingo-like version help students progress toward achieving standards 3 (construct viable arguments and critique the reasoning of others) and 6 (attend to precision, in particular the precision of mathematical descriptions). The Who Am I? puzzle, like all puzzles, also involves SMP 1, seeking an entry point and juggling multiple constraints and relationships.

Comment: Some of these statements are observably more difficult than others. "The ones digit is twice the tens digit" tends to be hard, initially, even for adults who fully know what "ones digit," "tens digit," and "twice" mean! Sorting out the **order** -- keeping track of which is twice which – takes a kind of attention we are not used to giving. Learning this is important in reading mathematics.

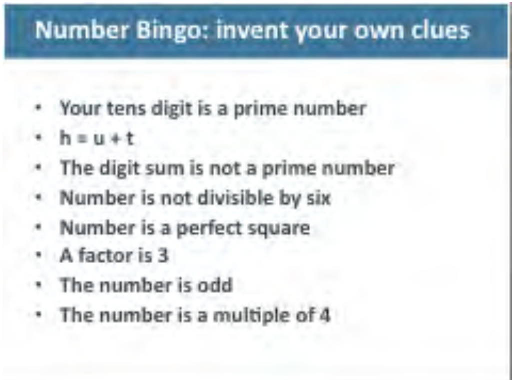
Slide 10



<u>You</u> can make up your own set of cards, using whatever clues fit the knowledge of the class or your goals for your class. Divisibility clues, odd-even, digits, symbols… Fit the kids and your goals.

BETTER YET, start with some examples so that your students get some ideas, and then HAVE YOUR STUDENTS ADD TO THE LIST OF POSSIBLE CLUES.

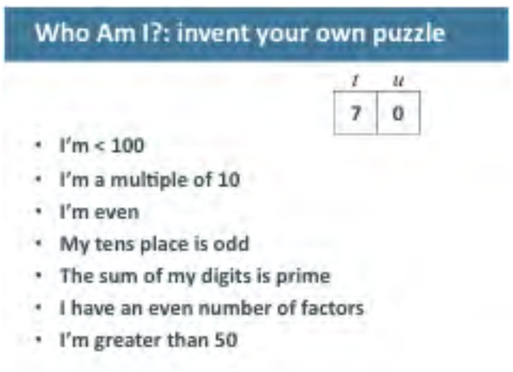
Slide 11



These clues were made up on the spot by attendees at the conference.

THIS MIGHT BE A STARTING LIST. Perhaps it could live on a poster that your students can add to over time.

Slide 12



This puzzle was made up on the spot by attendees at the conference.

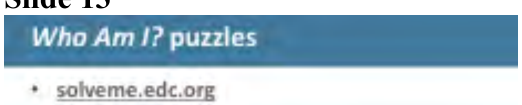
INVITE YOUR STUDENTS TO MAKE UP PUZZLES, TOO.

Start by choosing the answer, any number you like. With students, give a limit like two-digit or three-digit numbers until they get the idea that **a good puzzle** is one that is ***easy enough to solve*** and ***hard enough to be fun.*** Puzzles that are <u>too</u> complicated or too easy are not fun.

Then, after choosing the answer, write clues that are true about that answer. It's great (but not necessary!) to throw in unnecessary clues. For example, in the puzzle above, "I'm even" is unnecessary to say if we already know that the number is a multiple of 10. We also know that the number is less than 100 because we can see that it is a two-digit number, but the written form of the clue "I'm < 100" may help students learn the use of "<."

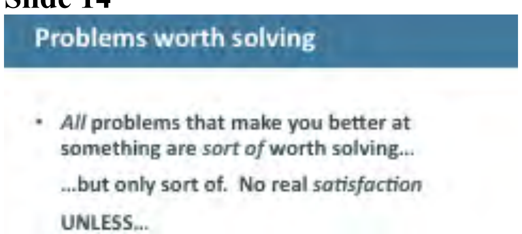
Comment: The language that makes it seem as if the number, itself, is speaking has no deep purpose other than fun, but it *does* help to "personalize" numbers. That is, it does suggest that numbers, like people or animals, have "personalities," characteristics that make them like or unlike other numbers we get to know well.

Slide 13

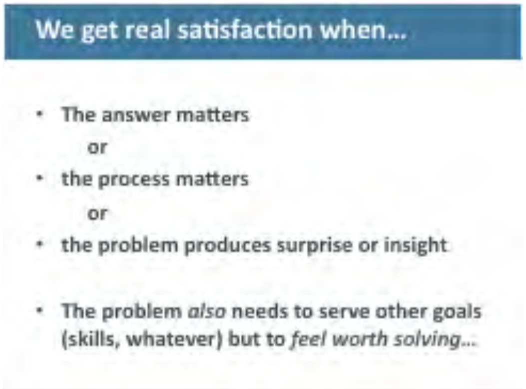


Who Am I? puzzles are currently in beta testing and will, within a very few months be "live" on solveme.edc.org.

Slide 14



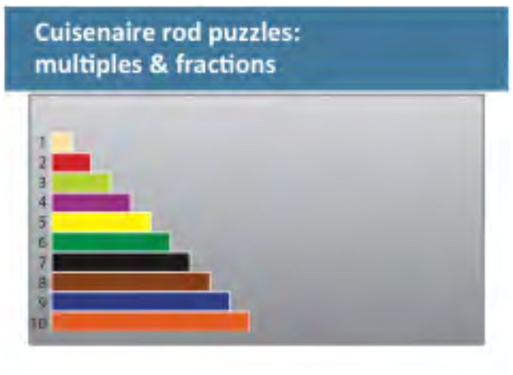
Slide 15



For most problems in textbooks, especially in elementary grades, the answer, itself, cannot possibly matter. We *do* care that students learn **how to figure out** how many marbles Miriam has—and they may care, too—but *"Miriam" isn't real*, and so the *answer* isn't what matters. And we don't care how much money Darryn makes for UNICEF if he buys 12 packs of cookies and 4 candy bars at one price and sells them in the school hall at some other price, because Darryn isn't our responsibility. We don't even know him!

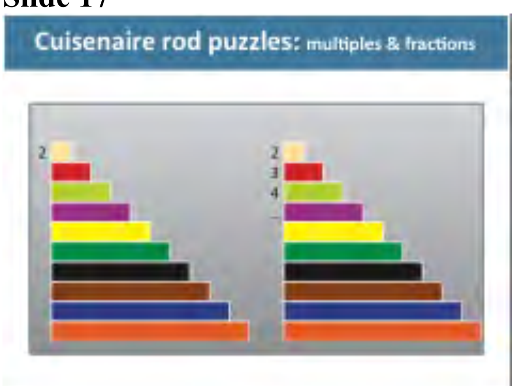
But my Chicago second graders (now over 50 years old!) did care very deeply how much *they* would make for UNICEF when they did these things because they were actually running that store in the hall and wanted to be sure they accumulated and retained the right amount of money! To them, the answer mattered a lot. The problem was real. And they came up with "real life puzzles" like figuring out "what a doorknob was worth" if they sold it. It became a deep discussion of whether other children were likely to want it, how much they would want it, whether they'd have a little or a lot of money, and whether the child who brought in the doorknob was willing to part with it for what it was likely to bring to UNICEF. This was all very deep mathematical (and economic) thinking.

Slide 16



The usual numbers associated with the rods.

Slide 17

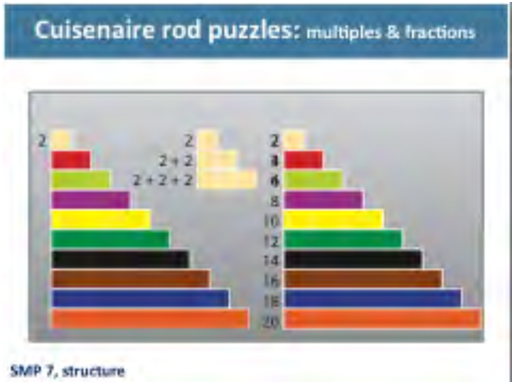


But if we let the white rod represent 2? What do the other rods represent?

For kids in grade 3, this might be worded more simply: What if we call the white rod 2? What should we call the other rods?

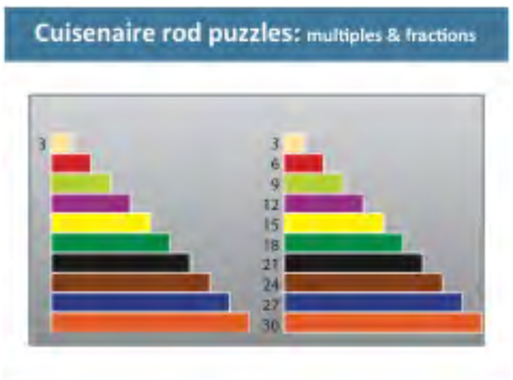
A first response is often just to continue the sequence starting at 2: 2, 3, 4,…

Slide 18



That *could* make sense, but violates the expectation that rods add together to match other rods. If white represents 2, then two whites represent 4.

Slide 19

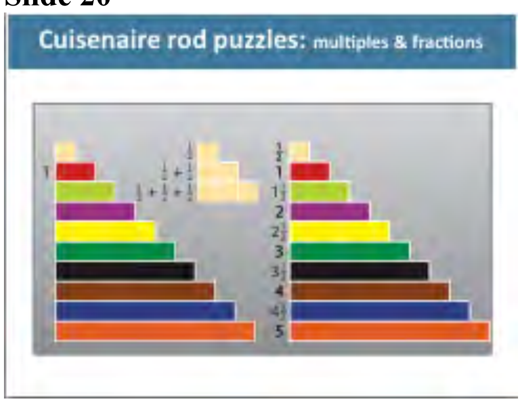


What if we call the white rod 3?

Same game generates multiples of 3.

This is "practice" but it also suggests some reasoning. For example, what number is represented by $r + y$ (the red rod plus the yellow rod)? We can simply add 9 + 15, but this also begins to build experience that we will <u>later</u> encode in this much more complicated way $3 \times 2 + 3 \times 5 = 3 \times (2 + 5) = 3 \times 7$.

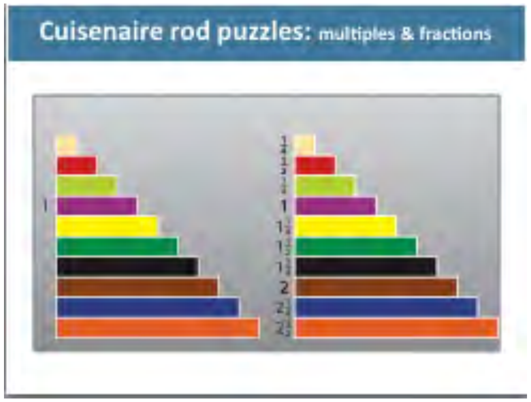
Slide 20



What if we call the *red* rod 1?

Though children pretty readily *get the idea*, they need to be *taught the notation*. Understanding the idea is relatively natural (partly because children already use the *language* of age—I'm eight and a half—already, even if they use it imprecisely), but the way the idea gets written is, like all writing, pure convention, just as knowing how to spell "weigh" and "way" is convention.

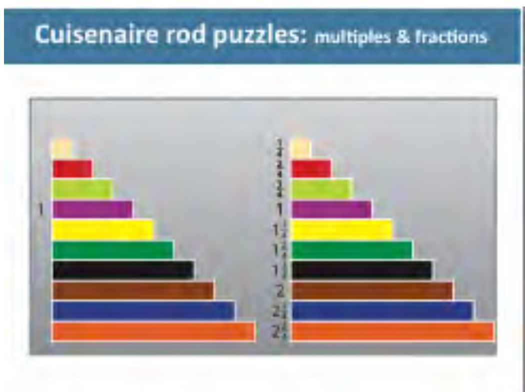
Slide 21



And what if we call the purple rod 1?

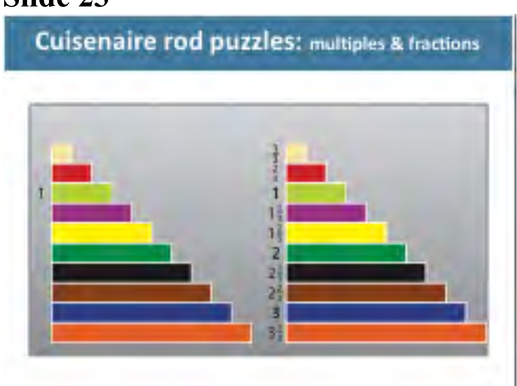
Some kids see the ½ and ¼ first, and continue from there.

Slide 22



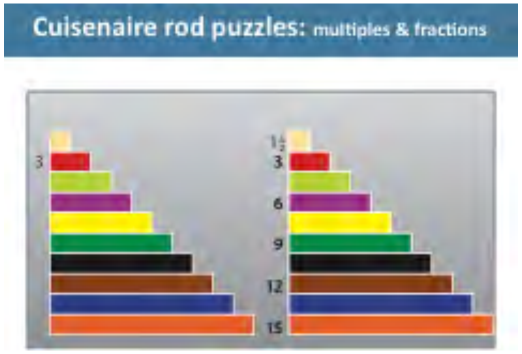
Some see the ¼ first, and then see two fourths and three fourths….

Slide 23



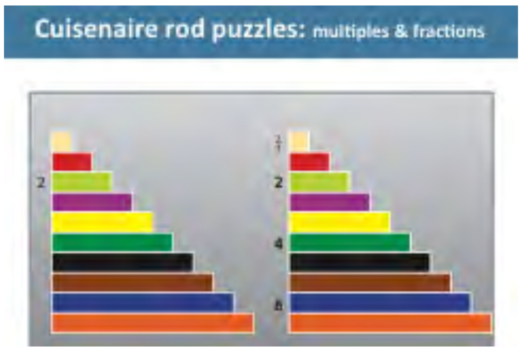
Thirds are harder, apparently, for most children, but when the idea and notation are taught, they can extend them.

Slide 24



Trickier! But a class of third graders unflatteringly described by their teacher as "low"—which was not at all *our* observation of them!—was perfectly delighted to figure this out, and succeeded.

Slide 25



This is hard! But once kids get it—the white rod is one third of that green rod, so one third of 2, which we write as 2/3—they love inventing puzzles of their own. What if the purple rod is 3? They are happily surprised to see results that feel quite familiar to them!

These puzzles appear in the K-5 comprehensive curriculum *Young Mathematicians Studio* (inspired by our work on *Think Math!*), the K-2 portion of which will pilot during 2015-2016, and in *Think Math!*, a published comprehensive K-5 curriculum. Contact mpi.edc.org about either of these programs. Puzzles of this kind also appear in *Transition to Algebra*, published by Heinemann. See TransitionToAlgebra.com for further information.

Slide 26

Another kind of puzzle. (To play with the puzzles, go to http://solveme.edc.org.)

Objects hang on a mobile. Sometimes you know the weight of the entire mobile (in the bottom-right example, the total weight is 28), and sometimes you know the weight of one or more of the objects (in that same mobile, the blue square weighs "1"), and sometimes (as in these examples), you know both. Your job is to figure out the weights of all the objects on the mobile.

solveme.edc.org has hundreds of mobiles of various challenge levels to solve, ***AND STUDENTS CAN MAKE UP <u>AND</u> <u>SHARE</u> THEIR OWN PUZZLES*** with their classmates, their parents, and the community of solvers in other schools.

Don't forget, also, to let students make nice puzzles with stickers on paper for posters in class! These third graders are proudly showing off their puzzles, one of which uses halves and fourths! The other puzzle has more than one solution even though the students have written only one on their "answer key." That's perfectly fine, too, because other students may find other answers or object that "the puzzle can't be solved," all of which makes for great debate in class!

Solving these puzzles gives a ton of raw arithmetic practice—already valuable on its own—but the puzzles give far more than that, as they begin to build strategies for solving equations that live on through algebra class. On the app, available for iPad, you can *drag* a beam off the puzzle to make an equation, and then drag matching items off both sides of the equation. You can also substitute the equations back into the mobile to help solve the puzzle. These are precisely the actions taught in algebra class, made accessible and logical to much younger learners.

Slide 27

Click on http://mathpractices.edc.org and have participants take the roles of Anita, Sam, and Dana and read the dialogue out loud.

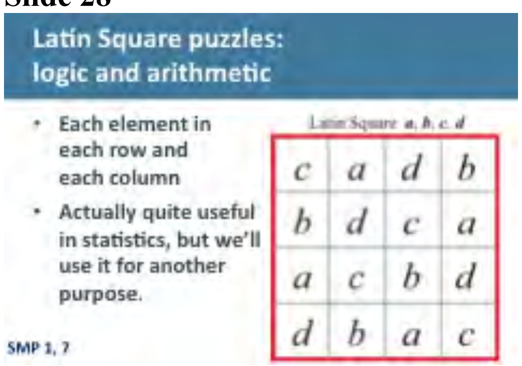
That dialogue is a crafted around purely fictional kids. It *deliberately* doesn't try to sound like your kids, use names like theirs, or model them in any way. But it does model several important mathematical habits of mind (representing the standards for mathematical practice). The three fictional kids in this fictional fifth/sixth grade class *have* been given "mathematical personalities." Anita is created to represent a pleasingly manic out-of-the-box thinker. Sam is more "normal-kid" but the asker of good questions. Sam's important contribution to mathematical dialogue *is* those questions; he shows he's hungry for ideas even when he's hungry for food, and he's curious enough to stick with an argument even when he doesn't yet know where it will lead. Dana might be the top math student who is sometimes satisfied knowing *how* even without knowing *why*, but then is also able to *use* the new knowledge she gains to make possibly unexpected leaps. The kids are real friends who get along well and are comfortable even with playful ribbing while they're trying to say something serious. The model for *your* students is not so much what they've figured out, but how they discuss mathematics. In some classes that regularly play with dialogues like these practicing for a few moments to themselves and then performing them out loud, students begin to "take on" favorite characters over time and—even from just play acting the dialogue, reading it out loud—will say things like "I feel so smart!"

Many dialogues of this kind can be found on mathpractices.edc.org, in the curriculum *Transition to Algebra* (http://www.heinemann.com/transitiontoalgebra/) and in *CME Project* mathematics (cmeproject.edc.org/), published by Pearson.

The chapter on "Thinking out loud" in *Making Sense of Algebra* gives practical guidance on how to conduct mathematical discussion in your classroom and also how to craft your own mathematical dialogues, similar to this one but covering whatever content you choose. We would love to see any dialogues you create, and will happily respond (and put up some of them as resources for others).

Making Sense of Algebra (described in detail at http://www.heinemann.com/products/E05301.aspx) is a professional book for teachers, aimed primarily at grades 6 through 9, but it presents stories from second and fourth grade classrooms as well, illustrating the meaning of the mathematical practice standards and ways to teach that develop those mathematical habits of mind. It describes all of the puzzles (except the Cuisenaire puzzles) mentioned in this presentation.

Slide 28

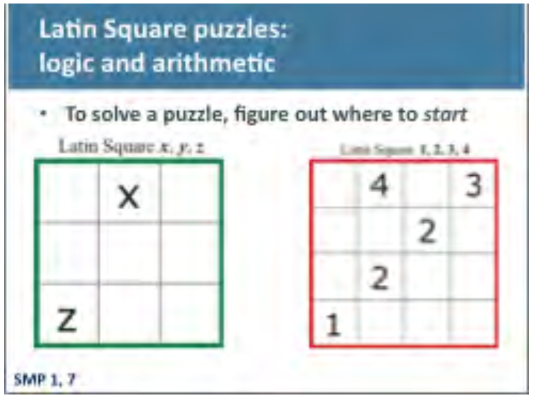


Looking up Latin Squares on wikipedia will tell you more than you probably want to know. ☺

Basically, an n by n Latin Square is a square array of n elements (which could be *anything*—letters, colors, numbers, kinds of candy) in which each row and each column contains every element (obviously, therefore, with no repeats). In this example, the elements are the letters a, b, c, and d, and each row and column contains all of them.

Latin Squares provide the *structure* that is necessary in many kinds of statistical designs for experiments that involve multiple varying conditions. Seeking and using that structure (mathematical practice standard #7) can be the basis for a variety of puzzles with useful content.

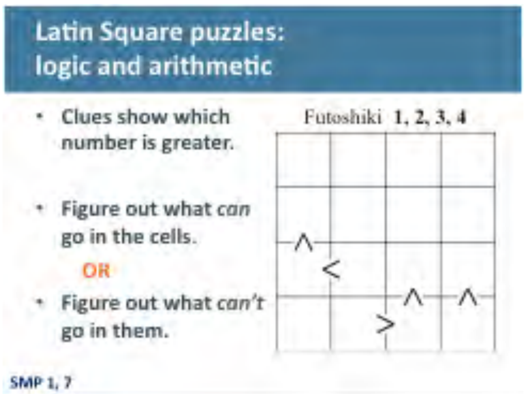
Slide 29



Start just with the Latin Square puzzles themselves. In the puzzle on the left, figure out where to put the remaining letters, *x*, *y*, and *z*, so that the solution is a Latin Square. Do the same on the right with the elements 1, 2, 3, and 4.

The key, as with most puzzles, is finding a good place to *start* (SMP #1) and using the structure well (SMP #7).

Slide 30



Again, the *solution* is a Latin square using only the numbers 1, 2, 3, and 4, but this time the *clues* are given only in the form of signs indicating which of two adjacent numbers is greater.

Here, a good way to seek an entry point is sometimes to think about what numbers *can't* go in certain locations. For example, in the bottom row, we know that the numbers in three of the cells are greater than the numbers in some other cell (in that row or in the row above). The number 1 cannot go in those locations, so it *must* go in the only remaining location because it must be *somewhere* in that row. You can find a location for 4 somewhere on the grid in the same way.

Kids who are taught about alligators with open mouths sometimes find that sufficient for memory, but many see the inequality sign as *pointing* to a number and that's all they remember about the alligator, thinking perhaps that's the number the alligator wants (nope!). Kids who are doing futoshiki puzzles get used to *using* the inequality sign to get information they want, information that *matters* to them. That's how we all learn language! And so it sticks.

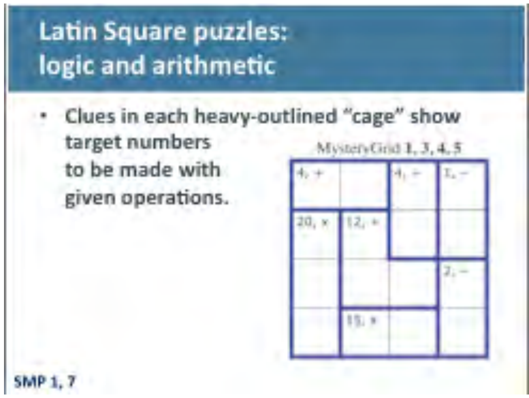
The logic of these can be wonderfully profound.

Note: futoshiki.org provides a huge (for all practical purposes, unlimited) set of futoshiki puzzles that you can print and use. You can select the size of the puzzle and also the difficulty level. The puzzles can be done on line, but they are in some ways more satisfying to do on paper.

They can get *very* difficult, so let students feel like experts before you push the challenge, and let *them* ask for harder puzzles. Even then, note that some will ask for the hardest (presumably to show how "smart" they are—see Dweck's work) and then be defeated. Give hard*er*, but not hard*est*—let them grow and enjoy the journey.

Slide 31

The clues can also involve arithmetic. This kind of puzzle was first introduced as KenKen puzzles, though the example shown here is a generalization of that puzzle type. In each heavily outlined region, called a "cage," is a **Target Number** and an **operation**. The numbers you write in the cells in that cage must make that target number using that operation. So, for example, in the top-left cage, you must make **7** using addition. The only way to do that with the allowed numbers (the title of this particular puzzle says you must use only 1, 3, 4, and 5) is to use 1 and 3. Ah! But we don't yet know which order to write them in! Should it be 1 and then 3 or 3 and then 1? We get a clue from the 20 cage in the left hand column. The three numbers in that cage must be multiplied and must then make 20. So 3 cannot go inside *that* cage and therefore it must be the top left cell.



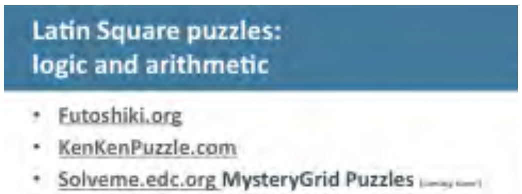
We don't (yet) know the order of the numbers in the <u>**20, ×**</u> cage, but we now know the order of the numbers in the <u>**4, +**</u> cage!.

Note: KenKenPuzzle.com provides a huge (for all practical purposes, unlimited) set of puzzles of this type that you can print and use. You can select the size of the puzzle and also the difficulty level. The puzzles can be done on line, but doing them on paper pushes more for the logic (and a better notation of that logic) than doing them on line. Also, the on-line interface includes a *timer* which works against the goal of thinking deeply about a puzzle. You can turn the timer off, but it's there and defaulted to *on*. Time pressure is threat, and we've already discussed how threat interferes with performance.

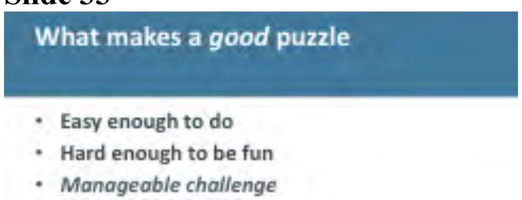
Like futoshiki, these puzzles, too, can get *very* difficult, so let students feel like experts before you push the challenge, and let *them* ask for harder puzzles. Note also that restricting the operations to + and – does not, by itself, make a puzzle easier. In fact, because there are fewer ways to make, say, 5, by multiplying than by adding (using numbers 1 through 5), having clues that use multiplication can make a puzzle easier.

The SolveMe.edc.org MysteryGrid Puzzles, currently in beta testing and soon to be live, will combine the features of all of these Latin Square puzzles. Like all of the SolveMe puzzles, it has the added important feature that students can make and share their own puzzles.

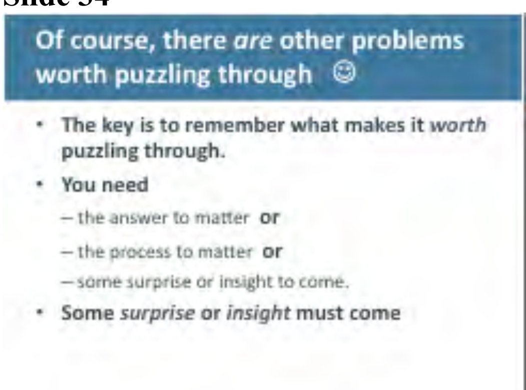
Slide 32



Slide 33



Slide 34



Slide 35